REAL ANALYSIS HW 7

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1. Problem 1

Consider the function $f := \sum_i \mathbf{1}_{E_i}$, where our E_i are the collection of sets having the property given in the problem statement. It is easy to see that f has the property that $f(x) \geqslant k$ for every $x \in [0,1]$ since we know that x belongs to at least k of our E_i. If we consider integrating f, we would also find that $\int_{[0,1]} f \geqslant km([0,1]) = k$.

Now, if we suppose for sake of contradiction that $m(E_i) < k/n$ for all $i = 1, \dots n$, then we'd also find that $k \leqslant \int_{[0,1]} f \leqslant \sum_i m(E_i) < k$, which is a clear contradiction. Thus there exists at least one E_j such that $m(E_j) \geqslant k/n$, as desired.

2. Problem 2

Without loss of generality, suppose that f is nonnegative since the general case follows from breaking f up into its positive and negative part. Then, $E_n := \{x \in E : \ f(x) > n\}$, with $E_\infty := \{x \in E : \ f(x) = \infty\}$, and since f is integrable, $\int_E f < M$ for some constant M. Then, as $E_n \subset E$, we have that $\int_{E_n} f \leqslant \int_E f$. Also, since $f > n$ on E_n, $\int_{E_n} f > nm(E_n)$. However, putting this together:

$$m(E_n) < \frac{M}{n}$$

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For all integers n. Letting $n \to \infty$, we see that $m(E_n) \to 0$. However it is clear that the set of E_n are a decreasing sequence tending to our desired set E_∞, and hence we can conclude that $m(E_\infty) = 0$, which was to be proved.

3. Problem 3

Suppose that f_n are nonnegative measurable functions. Then, $s_n := \sum_{i=1}^k f_k$ is an increasing sequence of measurable functions. For finite sums, it is clear that $\int_E s_n = \sum_{i=1}^n \int_E f_k$. By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_E s_n = \int_E \lim_{n\to\infty} s_n$$

And hence, combining the above:

$$\sum_{n=1}^{\infty} \int_E f_n = \int_E \sum_{n=1}^{\infty} f_n$$

As asserted.

4. Problem 4

(a). Define $A_n := \bigcup_{k=1}^n E_k$. Then, it is easy to see that $1_{A_n} = \sum_{i=1}^n 1_{E_i}$, and that $1_{A_n} \to 1_E$ as $n \to \infty$. We now have the conditions of the previous problem, since $\int_{A_n} f = \int_E f \cdot 1_{A_n} = \sum_{i=1}^n \int_E f \cdot 1_{E_i}$ and, since f is nonnegative and measurable, $f \cdot 1_{A_n}$ is an increasing sequence of measurable functions (tending to f), so that:

$$\sum_{n=1}^{\infty} \int_E f \cdot 1_{E_n} = \int_E \lim_{n\to\infty} f \cdot 1_{A_n}$$

And, since $\int_{E_n} f = \int_E f \cdot 1_{E_n}$:

$$\sum_{n=1}^{\infty}\int_{E_n} f = \int_E f$$

(b). Decompose $f = f^+ - f^-$ into its positive and negative parts. Then, by definition of integrability and the work of part (a) we know that $\sum_{n=1}^{\infty}\int_{E_n} f^+ = \int_E f^+$ and $\sum_{n=1}^{\infty}\int_{E_n} f^- = \int_E f^-$, so that

$$\int_E f = \int_E f^+ - \int_E f^- = \sum_{n=1}^{\infty}\int_{E_n} f^+ - \sum_{n=1}^{\infty}\int_{E_n} f^- = \sum_{n=1}^{\infty}\int_{E_n} f$$

So $\int_E f = \sum_{n=1}^{\infty}\int_{E_n} f$.

5. Problem 5

Consider first a simple function ϕ. Then, $\phi = \sum_i a_i \mathbf{1}_{A_i}$, and

$$\int_{[a-h,b-h]} \phi(x+h) = \sum_i a_i m(A_i \cap [a,b] - h) = \sum_i a_i m(A_i \cap [a,b]) = \int_{[a,b]} \phi(x)$$

Where we've used translation invariance of Lebesgue measure and the fact that $\mathbf{1}_A(x+h) = \mathbf{1}_{A-h}(x)$. Now let f be any arbitrary function. By the Simple approximation theorem, find sequences of simple functions ϕ_n, ψ_n increasing to f^+, f^-, respectively. Then, on one hand, $\int_{[a,b]} \phi_n \to \int_{[a,b]} f^+$, but by the above work, $\int_{[a,b]} \phi_n = \int_{[a-h,b-h]} \phi(x+h) \to \int_{[a-h,b-h]} f(x+h)$, and similarly for ψ_n. Since these limits must be unique, they are in fact equal. Hence, putting this all together:

$$\int_{[a-h,b-h]} f(x+h) = \int_{[a,b]} f(x)$$

So we are done.

6. Problem 6

Since $|f_n| \leqslant g_n$ for all n, we can take $n \to \infty$ to find that $|f| \leqslant g$. Then, consider:

$$\int_E \liminf_{n\to\infty} g + g_n - |f_n - f| \leqslant \liminf_{n\to\infty}(\int_E g + \int_E g_n) - \limsup_{n\to\infty} \int_E |f_n - f| \tag{6.1}$$
$$\implies \int_E 2g \leqslant \int_E 2g - \limsup_{n\to\infty} \int_E |f_n - f|$$
$$\implies \limsup_{n\to\infty} \int_E |f_n - f| \leqslant 0$$

Where we've employed Fatou's Lemma and the fact that $g_n \to g$ and $\int_E g_n \to \int_E g$. Thus, since $|f_n - f|$ is nonnegative and the limit superior of its integral is 0, we know that in fact $\lim_{n\to\infty} \int_E |f_n - f| = 0$ However,

$$\lim_{n\to\infty} |\int_E f_n - \int_E f| \leqslant \lim_{n\to\infty} \int_E |f_n - f| = 0$$

Hence $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$.

7. Problem 7

Note first that in order for $|\cos(f(x)\cdot\pi)| = 1$, we immediately find that $f(x) \in \mathbb{Z}$ by standard properties of trig functions. We also know that $|\cos(x)| \leqslant 1$, so we have a dominating function. We then have two cases: $\cos(f(x)\cdot\pi)| = 1$ or $|\cos(f(x)\cdot\pi)| < 1$. In the first case, $|\cos(f(x)\cdot\pi)|^n \to 1$, and in the second, $|\cos(f(x)\cdot\pi)|^n \to 0$. Then,

$$\lim_{n\to\infty} \int_{[0,1]} |\cos(f(x)\cdot\pi)|^n = \lim_{n\to\infty} \int_E |\cos(f(x)\cdot\pi)|^n + \lim_{n\to\infty} \int_{E^c} |\cos(f(x)\cdot\pi)|^n$$

By the Dominated Convergence Theorem, we can move the limit inside our integrals:

$$\lim_{n\to\infty}\int_{[0,1]}|\cos(f(x)\cdot\pi)|^n = \int_E 1 + \int_{E^c} 0 = m(E)$$

As asserted.